Applied Linear Systems (MIMO)

Course Notes

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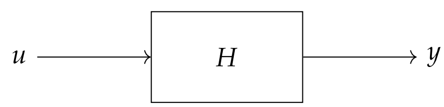
Contents

Chapter 1

Feedback Control and Basic Matrix Theory

1.1 Review of Classical Feedback Control

1.1.1 Control System Basics

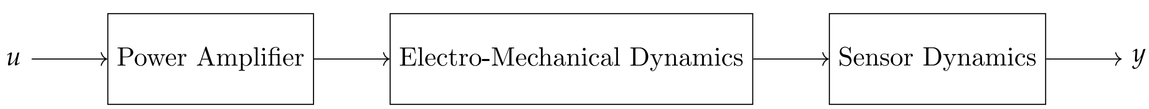


Single-Input Single-Output (SISO) System

Input u is the command (control) signal
Output y is the response (measurment) signal
Process (a.k.a. "Plant") H is a dynamic system that transforms the input into the output

Control Objective: Find u to generate a desired y.

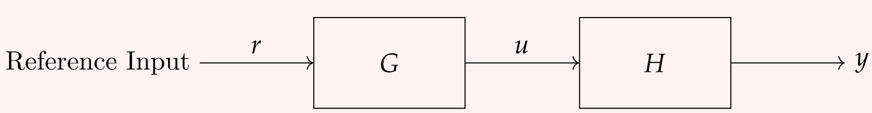
Typical electro-mechanical plants are made up of connected subsystems.



Single-Input Single-Output (SISO) System
with expanded plant (H)

Definition 1.1.1: Feedforward

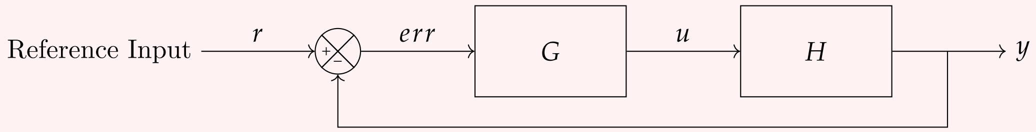
Feedforward control is a form of **open-loop** control (i.e. no explicit feedback loops) this is typically used for **command shaping** and/or **disturbance rejection**, and does not alter the dynamics of the process.



Feedforward system with feedforward controller G

Definition 1.1.2: Feedback Control

Feedback control is **closed-loop** control, where the contorl signal u is a function of the process output y. Feedback control is typically used for **regulation** or **tracking**, and changes the process's dynamics.



Feedback system

The transfer function of the unity feedback system above is:

$$T(s) = \frac{R(s)}{Y(s)} = \frac{G(s)H(s)}{1 + G(s)H(s)}$$

1.1.2 SIS0 Classical Control System Design

Theorem 1.1.1 CT Linear System

Process Modeling
Most continuous-time linear systems are represented in the Laplace domain as a ratio of polynomials:

$$H(s) = \frac{b_m s^m + ... + b_1 s^1 + b_0}{a_n s^n + ... + a_1 s^1 + a_0}, n \geqslant m$$

The numerator contains zeros (m-zeros)
The denominator contains poles (n-poles)

Note:-

The variable $s = \sigma + j\omega$ is the Laplace variable from the Laplace Transform:

$$H(s) = \int_0^\infty e^{-st} h(t) dt$$

and h(t) is the impulse response.

The controller is usually also represented by a transfer function $G(s)$ which is the ratio of polynomials,

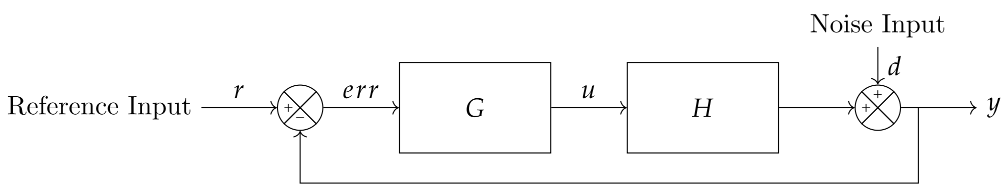
$$G(s) = \frac{U(s)}{E(s)}$$

where $E(s)$ is the error and $U(s)$ is the output of the controller.

1.1.3 Typical Design Problem

a. Design a controller such that the output $y(t)$ tracks the input $r(t)$.

b. The output is insensitive to measurment noise.



Feedback system

The output is given by:

$$Y(s) = H(s)U(s) + D(s) = H(s)[G(s)E(s)] + D(s)$$

the error is given by:

$$E(s) = R(s) - Y(s)$$

The closed-loop I/O relationship is given by

$$Y(s) = G(s)H(s)[R(s) - Y(s)] + D(s)$$
$$\Rightarrow Y(s) = \left(\frac{G(s)H(s)}{1 + G(s)H(s)}\right) R(s) + \left(\frac{1}{1 + G(s)H(s)}\right) D(s)$$

The coefficient of $R(s)$ is the Complementary Sensitivity $T(s)$
The coefficient of $D(s)$ is the Sensitivity Function $S(s)$

Note:-

We aim to keep the complementary sensitivity $T(s)$ high and sensitivity $S(s)$ low in an effort to minimize the effects of noise. In this way $Y(s) \approx R(s)$.

However turning up the gain too much causes instability.
This is the fundamental design tradeoff: as we increase gain of the compensator the sensitiviy to disturbances decreases (good), but also increases the chance of instability (bad).

LOOK INTO LOOP SHAPING

Designs are analyzed in the frequency domain for **Gain Margin** and **Phase Margin**:

- Bode Diagram: Mag & Phase vs. Freq
- Nyquist Diagram: Real vs. Imag
- Nichols Chart: Magnitude vs. Phase

Definition 1.1.3: Common SISO Controllers

PID Compensator

$$G_{PID}(s) = k_p + k_i \frac{1}{s} + k_d s$$

Lead-Lag Compensator

$$G_{LL}(s) = k \frac{s + b}{s + a}$$

$$a > b \longrightarrow \text{Lead}$$
$$a < b \longrightarrow \text{Lag}$$

1.2 Review Vector/Matrix Theory

1.2.1 Matrix Operations

Addition

$$C_{[NxM]} = A_{[NxM]} + B_{[NxM]}$$

Matrix addition is an element-by-element operation.

$$C = A + B \Leftrightarrow c_{ij} = a_{ij} + b_{ij}$$
$$\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

Multiplication

$$C_{[NxM]} = A_{[NxP]} * B_{[PxM]}$$

Matrix multiplication is the result of dot products between rows and columns.

$$\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} c_{11} * b_{11} + c_{12} * b_{21} & c_{11} * b_{12} + c_{12} * b_{22} \\ c_{21} * b_{11} + c_{22} * b_{21} & c_{21} * b_{12} + c_{22} * b_{22} \end{bmatrix}$$

Identity Matrix

Identity matricies have ones along the diagonal and zero everywhere else.

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

if A is of equal dimension as I,

$$A = IA$$

Symetric Matrix

A symetric matrix is one in which the transpose of the matrix is equal to the original matrix.

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

Inverse Matrix

A square matrix A has an inverse if a matrix B can be found such that:

$$BA = I$$

The B matrix is the inverse of A,

$$B = A^{-1}$$

This implies $\Rightarrow AA^{-1} = I$ and $A^{-1}A = I$
A general 2×2 matrix has the inverse:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \left(\frac{1}{ad - bc}\right) \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Determinant

The determinant of a matrix A is a scalar function of matrix components,
for a 2×2 matrix:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = (ad - bc)$$

for a 3×3 matrix:

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$|A| = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$|A| = a(ei - fh) - b(di - fg) + c(dh - eg)$$

1.2.2 Eigenvalue Problem

One of the most important problems in matrix analysis is the Eigenvalue Problem.

Definition 1.2.1: The Eigenvalue Problem

Given an $N \times N$ matrix, A, find the scalar(s) λ and vector(s) v that satisfy:

$$A_{[N\times N]}v_{[N\times 1]} = \lambda_{[1\times 1]}v_{[N\times 1]}$$

Of course we could set $v = 0$ but this does not tell us much about the system.

Theorem 1.2.1 Finding Eigenvalues and Eigenvectors

Instead, we assume $(A - \lambda I)^{-1}$ does not exist, which is equivalent to assuming that the determinant of $(A - \lambda I)$ is equal to zero.

$$|A - \lambda I| = 0$$

This yields an N^{th}-order **characteristic polynomial** in λ,

$$\lambda^N + c_{N-1}\lambda^{N-1} + \ldots + c_1\lambda^1 + c_0 = 0$$

The solution (roots) of this polynomial are the N **eigenvalues** of A. The eigenvalues are denoted as:

$$\lambda_n,\ n = 1, \ldots, N$$

Each eigenvalue has an associated **eigenvector** v_n that is the solution of:

$$(A - \lambda_n I)v_n = 0$$

Note:-

If the real part of all the eigenvalues are negative the system is stable. If even one eigenvalue is positive the system is unstable.

Chapter 2

State Space Modeling and Dynamic Response of Linear Systems

2.0.1 State Representations

In state-space a mathematical model is required to develop a controller.

Definition 2.0.1: State of a System

The State is a set of signals which, along with the current time, summersizes the current configuration of the dynamic system.

States can be positions, velocities, accelerations, forces, momentum, torques, voltage, current, charge, etc.

Note:-

General ideas for understanding state-space modeling:

- The choice of state variables is not unique
- A model using a specific choice of state variables is called **Realization**
- A system has infinitely many realizations
- the **Dimension** of realization is = the number of states in that realization
- The order of a system is the minimal number of state variables required to descirbe it ($Order \leqslant Dimensions$)

Inputs

- **Endogenous Inputs** Signals generated from inside the dynamic system.
- **Exogenous Inputs** Signals generated outside the dynamic system.

Outputs
Signals we measure for feedback or quantifying performance.
Outputs can be written as linear combinations of Inputs and States.

The standard notation convention for a state-space representation requires two parts:

State Equation

$$\dot{x}_{[N\times 1]} = A_{[N\times N]}x_{[N\times 1]} + B_{[N\times M]}u_{[M\times 1]}$$

Output Equation

$$y_{[P\times 1]} = C_{[P\times N]}x_{[N\times 1]} + D_{[P\times M]}u_{[M\times 1]}$$

Conventional Nomenclature

$x(t) \longrightarrow$ State Vector
$A \longrightarrow$ State Matrix
$B \longrightarrow$ Input Matrix
$u(t) \longrightarrow$ Input Vector
$y(t) \longrightarrow$ Output Vector
$C \longrightarrow$ Output Matrix
$D \longrightarrow$ Direct Transmission Matrix (Feedthrough)

First-Order Equations

Models of dynamic processes are written as differential equtions with time t as the independent variable.

For mechanical systems, we typically have:
$x(t)$ = Position or Displacement
$\dot{x}(t)$ = Velocity
$\ddot{x}(t)$ = Acceleration

Example 2.0.1 (State Space Realization of a DC Motor)

The dynamic equations are:

$$J\frac{d^2\theta(t)}{dt^2} + b\frac{d\theta(t)}{dt} = K_T i(t)$$

$$L\frac{di(t)}{dt} + Ri(t) + K_V\frac{d\theta(t)}{dt} = V_{in}(t)$$

The First equation describes the mecahanical dynamics of a DC motor and the second equation describes the electrical dynamics.
Definition of Terms:

$J\frac{d^2\theta(t)}{dt^2} \longrightarrow$ Applied Torque

$b\frac{d\theta(t)}{dt} \longrightarrow$ Rotational Damping

$K_T i(t) \longrightarrow$ Coupling between mechanical torque constant K_T and electrical current $i(t)$

$L\frac{di(t)}{dt} \longrightarrow$ Voltage across coil (inductor)

$Ri(t) \longrightarrow$ Voltage drop across internal resistance

$K_V\frac{d\theta(t)}{dt} \longrightarrow$ Back EMF (coupling mechanical and electircal)

$V_{in}(t) \longrightarrow$ Applied torque

After we have our dynamic equation we can create ”candidate states”:

$$x_1 = i(t),\ x_2(t) = \theta(t),\ x_3(t) = \dot{\theta}(t)$$

ALWAYS choose the lowest possible order derivative

Thus we have,

$$J\dot{x}_3 + bx_3 = K_T x_1$$

$$L\dot{x}_1 + Rx_1 + K_V x_3 = V_{in}(t)$$

To complete our system we need one first order ODE for each "candidate state"

$$\dot{x}_1 = -\frac{R}{L}x_1 + 0x_2 - \frac{K_V}{L}x_3 + \frac{1}{L}V_{in}$$

$$\dot{x}_2 = 0x_1 + 0x_2 + 1x_3 + 0$$

$$\dot{x}_3 = \frac{K_T}{J}x_1 + 0x_2 - \frac{b}{J}x_3 + 0$$

Here x's are our states and the V_{in} term is our input.
Now, a state-space equation realization can be constructed,

State Equation:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & 0 & -\frac{K_V}{L} \\ 0 & 0 & 1 \\ \frac{K_T}{J} & 0 & -\frac{b}{J} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ 0 \\ 0 \end{bmatrix} V_{in}$$

If we have a sensor measuring angular velocity $\dot{\theta}(t)$.

Output Equation:

$$\begin{bmatrix} y \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} V_{in}$$

Because all coefficients of $\dot{x}_2$ are all equal to 0, it does not contribute. Thus, we can remove it.
Now, we are left with:

State Equation:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & -\frac{K_V}{L} \\ \frac{K_T}{J} & -\frac{b}{J} \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} V_{in}$$

Output Equation:

$$\begin{bmatrix} y \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} V_{in}$$

On the other hand , if we had a sensor that measured angular position $\theta(t)$ we could not remove x_2.
$\rightarrow$ Removing states that do not contribute, gives us what is called the **minimum realization**.

2.1 Solving General Differential Equations

INCLUDE NOTES FROM MISSED LECTURE
$\Rightarrow$ MATRIX EXPONENTIAL

LTI Solution VII

Continuing we havem

$$x(t) = e^{At}c$$

what is the constant vector c? Assume we know the state at inital time $t = t_0$:

LTI OSLUTION VIII

Substituting into assumed solution, we find the homogeneous solution to the state-space **Initial Value Problem** is:

$$x(t) = e^{At}c = e^{A(t-t_0)}x(t_0) = \Phi(t, t_0)x(t_0)$$

The **State Transition Matrix** defines how the state evolves from its inital conditions:

$$e^{A(t-t_0)} = \Phi(t, t_0)$$

Note:-

the state transition matrix is always a function of the difference between time t and a different time t_0
For LTI systems, we can always set this inital time equal to 0 (Not true for LTV systems).

INCLUDE LTI SOLUTION X
Useful Properties of the Transition Matrix

Example 2.1.1 (LTI System Solution)

Let the state equations be:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 0 & -10 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

The matrix exponential for this system is:

$$e^{At} = \begin{bmatrix} e^{-3t} & \frac{1}{7}(e^{-3t} - e^{-10t}) \\ 0 & e^{-10t} \end{bmatrix}$$

(discussion of how to obtain e^{At} in later sections)
Now we find the state response to the following step input

$$u(t) = \begin{cases} 0 & t < 0 \\ 5 & t \geqslant 0 \end{cases}$$

and the following input conditions:

$$x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

First compute the response due to inital conditions:

$$e^{At}x(0) = \begin{bmatrix} e^{-3t} & \frac{1}{7}(e^{-3t} - e^{-10t}) \\ 0 & e^{-10t} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{8}{7}e^{-3t} - \frac{1}{7}(e^{-10t}) \\ e^{-10t} \end{bmatrix}$$

Next, compute the forced response using the second form of the convolution integral:

$$\int_{t_0}^{t} [e^{At}B]u(t - \tau)d\tau$$

Chapter 3

Frequency-Domain Analysis

Chapter 4

Controllability, Observability, and Pole Placement Design

Chapter 5

LQG/LQR Optimal Control